Topic - Problem based on Symmetric functions of the roots B.Sc. I

1. If α, β, γ be the roots of the cubic $a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$, then find the value of

(i) $\Sigma(\alpha-\beta)^2$ (ii) $(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)(2\gamma-\alpha-\beta)$

Solution → Given cubic equation

$$a_0x^3 + 3a_1x^2 + 3a_2x + a_3 = 0$$

Also α, β, γ are its roots.

Now we have $\alpha+\beta+\gamma = \Sigma\alpha = (-1)^1 \frac{3a_1}{a_0} = -\frac{3a_1}{a_0}$

$\alpha\beta+\beta\gamma+\gamma\alpha = \Sigma\alpha\beta = (-1)^2 \frac{3a_2}{a_0} = \frac{3a_2}{a_0}$

$\alpha\beta\gamma = (-1)^3 \frac{a_3}{a_0} = -\frac{a_3}{a_0}$

Now $\Sigma(\alpha-\beta)^2 = (\alpha-\beta)^2 + (\beta-\gamma)^2 + (\gamma-\alpha)^2$

$$= \alpha^2+\beta^2-2\alpha\beta+\beta^2+\gamma^2-2\beta\gamma+\gamma^2+\alpha^2-2\gamma\alpha$$

$$= 2(\alpha^2+\beta^2+\gamma^2) - 2(\alpha\beta+\beta\gamma+\gamma\alpha)$$

$$= 2\Sigma\alpha^2 - 2\Sigma\alpha\beta$$

$$= 2\left[(\Sigma\alpha)^2 - 2\Sigma\alpha\beta\right] - 2\Sigma\alpha\beta$$

$$= 2(\Sigma\alpha)^2 - 6\Sigma\alpha\beta$$

$$= 2\left(-\frac{3a_1}{a_0}\right)^2 - 6\left(\frac{3a_2}{a_0}\right)$$

$$= \frac{18a_1^2}{a_0^2} - \frac{18a_2}{a_0}$$

$$= \frac{18a_1^2 - 18a_2a_0}{a_0^2}$$

$$= 18\left[\frac{a_1^2 - a_2a_0}{a_0^2}\right]$$

(ii) $(2\alpha-\beta-\gamma)(2\beta-\gamma-\alpha)(2\gamma-\alpha-\beta)$

$$= [3\alpha-(\alpha+\beta+\gamma)][3\beta-(\alpha+\beta+\gamma)][3\gamma-(\alpha+\beta+\gamma)]$$

$$= [3\alpha - \Sigma\alpha][3\beta - \Sigma\alpha][3\gamma - \Sigma\alpha]$$

$$= \left(3\alpha + \frac{3a_1}{a_0}\right)\left(3\beta + \frac{3a_1}{a_0}\right)\left(3\gamma + \frac{3a_1}{a_0}\right)$$

$$= \left[9\alpha\beta + \frac{9a_1}{a_0}\alpha + \frac{9a_1}{a_0}\beta + \frac{9a_1^2}{a_0^2}\right]\left[3\gamma + \frac{3a_1}{a_0}\right]$$

$$= 27\alpha\beta\gamma + 27\frac{a_1}{a_0}\alpha\beta + \frac{27a_1}{a_0}\alpha\gamma + \frac{27a_1^2}{a_0^2}\alpha + \frac{27a_1}{a_0}\beta\gamma + \frac{27a_1^2}{a_0^2}\beta + \frac{27a_1^2}{a_0^2}\gamma + 27\frac{a_1^3}{a_0^3}$$

$$= 27\left[\alpha\beta\gamma + \frac{a_1}{a_0}(\alpha\beta + \beta\gamma + \gamma\alpha) + \frac{a_1^2}{a_0^2}(\alpha+\beta+\gamma) + \frac{a_1^3}{a_0^3}\right]$$

$$= 27\left[-\frac{a_3}{a_0} + \frac{a_1}{a_0} \times \frac{3a_2}{a_0} + \frac{a_1^2}{a_0^2}\left(-\frac{3a_1}{a_0}\right) + \frac{a_1^3}{a_0^3}\right]$$

$$= 27\left[-\frac{a_3}{a_0} + \frac{3a_1a_2}{a_0^2} - \frac{3a_1^3}{a_0^3} + \frac{a_1^3}{a_0^3}\right]$$

$$= 27\left[\frac{-a_0^2a_3 + 3a_1a_2a_0 - 2a_1^3}{a_0^3}\right]$$

$$= -\frac{27}{a_0^3}\left[2a_1^3 - 3a_1a_2a_0 + a_0^2a_3\right]$$

* for the Cubic Equation $x^3 + px^2 + qx + r = 0$ Find the value of the symmetric functions

(i) $\Sigma\dfrac{\alpha^2 + \beta\gamma}{\beta+\gamma}$ (ii) $\Sigma(\alpha+\beta-\gamma)^3$

(iii) $\Sigma\left(\dfrac{\alpha-\beta}{\alpha+\beta}\right)^2$ (iv) $\Sigma\dfrac{\beta^2+\gamma^2}{\beta+\gamma}$

(v) $\Sigma\dfrac{\alpha^2+\beta^2}{\alpha\beta}$

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Solution :→

Here Given Equation is

$$x^3 + px^2 + qx + r = 0$$

Since it is Cubic Equation, and α, β, γ are its roots.

then we have

$$\Sigma\alpha = \alpha + \beta + \gamma = -p$$

$$\Sigma\alpha\beta = \alpha\beta + \beta\gamma + \gamma\alpha = q$$

$$\alpha\beta\gamma = -r$$

① We have to find the value of

$$\Sigma \frac{\alpha^2 + \beta\gamma}{\beta + \gamma} = \frac{\alpha^2 + \beta\gamma}{\beta + \gamma} + \frac{\beta^2 + \gamma\alpha}{\gamma + \alpha} + \frac{\gamma^2 + \alpha\beta}{\alpha + \beta}$$

$$= \frac{(\alpha^2 + \beta\gamma)(\alpha+\beta)(\gamma+\alpha) + (\beta^2 + \gamma\alpha)(\beta+\gamma)(\alpha+\beta) + (\gamma^2 + \alpha\beta)(\beta+\gamma)(\gamma+\alpha)}{(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)}$$

$$= \frac{\Sigma(\alpha^2 + \beta\gamma)(\gamma+\alpha)(\alpha+\beta)}{(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)}$$

$$N^r = \Sigma(\alpha^2 + \beta\gamma)(\gamma+\alpha)(\alpha+\beta)$$

$$= \Sigma(\alpha^2 + \beta\gamma)[\alpha\gamma + \beta\gamma + \alpha\beta + \alpha^2]$$

$$= \Sigma(\alpha^2 + \beta\gamma)(\alpha^2 + q) \qquad \text{As } \Sigma\alpha\beta = q$$

$$= \Sigma\alpha^4 + q\Sigma\alpha^2 + \Sigma\alpha^2\beta\gamma + q\Sigma\beta\gamma.$$

Now $\Sigma\alpha^4 = \alpha^4 + \beta^4 + \gamma^4$

$$= (\alpha^2 + \beta^2 + \gamma^2)^2 - 2[\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2]$$

$$= (\Sigma\alpha^2)^2 - 2\Sigma\alpha^2\beta^2$$

Now $\Sigma\alpha^2\beta^2 = (\Sigma\alpha\beta)^2 - 2\Sigma\alpha^2\beta\gamma$

$$= q^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)$$

$$= q^2 - 2(-p)(-r)$$

$$= q^2 - 2pr.$$

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$\therefore \sum \alpha^4 = (\sum \alpha^2)^2 - 2 \sum \alpha^2\beta^2$

$$\sum \alpha^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$
$$= p^2 - 2q$$
$$(\sum \alpha^2)^2 = (p^2 - 2q)^2$$

$$\therefore \sum \alpha^4 = (\sum \alpha^2)^2 - 2 \sum \alpha^2\beta^2$$
$$= (p^2 - 2q)^2 - 2(q^2 - 2pr)$$

Hence $N^r = \cancel{\sum \alpha^4 + q \sum \alpha^2}$

$$(p^2 - 2q)^2 - 2(q^2 - 2pr) + q(p^2 - 2q) + (-r)(-p) + q^2$$
$$= p^4 - 4p^2q + 4q^2 - 2q^2 + 4pr + p^2q - 2q^2 + pr + q^2$$
$$= p^4 - 3p^2q + 5pr + q^2$$

$$D^r = (\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)$$
$$= (\alpha+\beta+\gamma-\gamma)(\alpha+\beta+\gamma-\alpha)(\alpha+\beta+\gamma-\beta)$$
$$= (\sum\alpha - \gamma)(\sum\alpha - \alpha)(\sum\alpha - \beta)$$
$$= (-p-\gamma)(-p-\alpha)(-p-\beta)$$
$$= -(p+\alpha)(p+\beta)(p+\gamma)$$
$$= -[p^2 + p\beta + p\alpha + \alpha\beta][p+\gamma]$$
$$= -[p^3 + p^2\gamma + p^2\beta + p\beta\gamma + p^2\alpha + p\alpha\gamma + p\alpha\beta + \alpha\beta\gamma]$$
$$= -[p^3 + p^2(\alpha+\beta+\gamma) + p(\alpha\beta + \beta\gamma + \gamma\alpha) + \alpha\beta\gamma]$$
$$= -[\cancel{p^3} - \cancel{p^3} + pq - r]$$
$$= r - pq$$

Hence $\sum \frac{\alpha^2 + \beta\gamma}{\beta+\gamma} = \frac{N^r}{D^r} = \frac{p^4 - 3p^2q + 5pr + q^2}{r - pq}$

(ii) $\Sigma(\alpha+\beta-\gamma)^3$

$$= (\alpha+\beta-\gamma)^3 + (\beta+\gamma-\alpha)^3 + (\gamma+\alpha-\beta)^3$$

$$= [\alpha+\beta+\gamma-2\gamma]^3 + [\alpha+\beta+\gamma-2\alpha]^3 + [\alpha+\beta+\gamma-2\beta]^3$$

$$= [\Sigma\alpha - 2\gamma]^3 + (\Sigma\alpha - 2\alpha)^3 + (\Sigma\alpha - 2\beta)^3$$

$$= -(p+2\gamma)^3 - (p+2\alpha)^3 + (p+2\beta)^3$$

$$= -[(p+2\gamma)^3 + (p+2\alpha)^3 + (p+2\beta)^3]$$

$$= -[p^3 + 6p^2\gamma + 12p\gamma + 8\gamma^3 + p^3 + 6p^2\alpha + 12p\alpha + 8\alpha^3 + p^3 + 6p^2\beta + 12p\beta + 8\beta^3]$$

$$= -[3p^3 + 6p^2(\alpha+\beta+\gamma) + 12p(\alpha+\beta+\gamma) + 8(\alpha^3+\beta^3+\gamma^3)]$$

$$= -[3p^3 + 8\Sigma\alpha^3 + 6p^2\Sigma\alpha + 12p\Sigma\alpha]$$

$$= -[3p^3 + 8(3pq - p^3 - 3r) + 6p^2(-p) + 12p(p^2-2q)]$$

$$= -[3p^3 + 24pq - 8p^3 - 8r - 6p^3 + 12p^3 - 24pq]$$

$$= -[p^3 - 24r]$$

$$= 24r - p^3$$

(iii) $\Sigma\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2 = \frac{(\alpha-\beta)^2}{(\alpha+\beta)^2} + \frac{(\beta-\gamma)^2}{(\beta+\gamma)^2} + \frac{(\gamma-\alpha)^2}{(\gamma+\alpha)^2}$

$$= \frac{(\alpha+\beta)^2 - 4\alpha\beta}{(\alpha+\beta)^2} + \frac{(\beta+\gamma)^2 - 4\beta\gamma}{(\beta+\gamma)^2} + \frac{(\gamma+\alpha)^2 - 4\gamma\alpha}{(\gamma+\alpha)^2}$$

$$= 1 - 4\left[\frac{\alpha\beta}{(\alpha+\beta)^2}\right] + 1 - \frac{4\beta\gamma}{(\beta+\gamma)^2} + 1 - \frac{4\gamma\alpha}{(\gamma+\alpha)^2}$$

$$= 3 - 4\left[\frac{\alpha\beta}{(\alpha+\beta)^2} + \frac{\beta\gamma}{(\beta+\gamma)^2} + \frac{\gamma\alpha}{(\gamma+\alpha)^2}\right]$$

$$= 3 - 4\sum\frac{\alpha\beta}{(\alpha+\beta)^2}$$

$$= 3 - 4\left[\frac{\alpha\beta(\beta+\gamma)^2(\gamma+\alpha)^2 + \beta\gamma(\gamma+\alpha)^2(\alpha+\beta)^2 + \gamma\alpha(\alpha+\beta)^2(\gamma+\beta)^2}{[(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)]^2}\right]$$

$$= 3 - \frac{4}{[(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)]^2}\sum\alpha\beta(\beta+\gamma)^2(\gamma+\alpha)^2 \quad \text{——— } (1)$$

$$(\beta+\gamma)^2(\gamma+\alpha)^2 = [\gamma^2 + \alpha\beta+\beta\gamma+\gamma\alpha]^2$$

$$= (\gamma^2 + \sum\alpha\beta)^2 = (\gamma^2+q)^2$$

$$= \gamma^4 + 2\gamma^2 q + q^2$$

$$\therefore \sum\alpha\beta(\beta+\gamma)^2(\gamma+\alpha)^2 = \sum\alpha\beta(\gamma^4 + 2\gamma^2 q + q^2)$$

$$= \sum\alpha\beta\gamma^4 + 2q\sum\alpha\beta\gamma^2 + q^2\sum\alpha\beta.$$

$$= [\alpha\beta\gamma^4 + \alpha\beta^4\gamma + \alpha^4\beta\gamma] + 2q[\alpha\beta\gamma^2 + \beta\gamma\alpha^2 + \gamma\alpha\beta^2]$$

$$+ q\sum\alpha\beta$$

$$= \alpha\beta\gamma(\alpha^3+\beta^3+\gamma^3) + 2q\alpha\beta\gamma(\alpha+\beta+\gamma) + q^2\sum\alpha\beta$$

$$= \alpha\beta\gamma\sum\alpha^3 + 2q\alpha\beta\gamma\sum\alpha + q^2\sum\alpha\beta$$

$$= -r(3pq - p^3 - 3r) + 2q(-p)(-r) + q^2 q$$

$$= -3pqr + p^3 r + 3r^2 + 2pqr + q^3$$

$$= -pqr + p^3 r + 3r^2 + q^3$$

Also $B^2 = \left[(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)\right]^2$

$$= \left[(\Sigma\alpha-\gamma)(\Sigma\alpha-\alpha)(\Sigma\alpha-\beta)\right]^2$$

$$= \left[[-p-\gamma][-p-\alpha](-p-\beta)\right]^2$$

$$= \left[(p+\alpha)(p+\beta)(p+\gamma)\right]^2$$

$$= \left[p^3+p^2(\alpha+\beta+\gamma)+p(\alpha\beta+\beta\gamma+\gamma\alpha)+\alpha\beta\gamma\right]^2$$

$$= \left[p^3+p^2\Sigma\alpha+p\Sigma\alpha\beta-r\right]^2$$

$$= \left[p^3-p^3+pq-r\right]^2$$

$$= (pq-r)^2$$

$$\therefore \prod\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2 = 3-\frac{4}{(pq-r)^2}\times(q^3+3r^2+p^3r-pqr)$$

$$= \frac{3(pq-r)^2-4(q^3+3r^2+p^3r-pqr)}{(pq-r)^2}$$

$$= \frac{3\left[p^2q^2-2pqr+r^2\right]-4(q^3+3r^2+p^3r-pqr)}{(pq-r)^2}$$

$$= \frac{3p^2q^2-6pqr+3r^2-4q^3-12r^2-4p^3r+4pqr}{(pq-r)^2}$$

$$= \frac{3p^2q^2+3r^2-6pqr-4q^3-12r^2-4p^3r+4pqr}{(pq-r)^2}$$

$$= \frac{3p^2q^2-9r^2-2pqr-4q^3-4p^3r}{(pq-r)^2}$$